ON THE NUMERICAL INVERSION OF LAPLACE AND MELLIN TRANSFORMS

by

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INTRODUCTION

The Laplace transform of a function $f(x)$ is defined to be

$$\varphi(t) = \int_0^\infty e^{-tx} f(x)\, dx.$$

If $f(x).e^{-bx}$ is bounded for some positive b, then $\varphi(t)$ exists for $R(t) > b$ and it can be shown [1] that the transform $f \to \varphi$ has the inverse:

$$f(x) = 1/2\pi i \int_{b'-i\infty}^{b'+i\infty} e^{xt} \varphi(t)\, dt, \quad \text{where } b' > b.$$

Similarly the Mellin transform of a function $f(x)$ is defined to be

$$\Psi(t) = \int_0^\infty x^t f(x)\, dx$$

$$= \int_0^1 x^t f(x)\, dx + \int_1^\infty x^t f(x)\, dx$$

$$= \Psi_1(t) + \Psi_2(t) \qquad \text{say.}$$

Let
$$f_1(x) = f(x) \qquad x < 1$$
$$= 0 \qquad x > 1$$
$$f_2(x) = 0 \qquad x < 1$$
$$= f(x) \qquad x > 1.$$

Then $\Psi_1(t) = \int_0^1 x^t f_1(x)\, dx$

Putting $x = e^{-u}$ we have

$$\Psi_1(t) = \int_0^\infty e^{-ut} f_1(e^{-u}) e^{-u}\, du$$

So that $\Psi_1(t)$ is the Laplace transform of $f_1(e^{-u})\, e^{-u}$ and $\Psi_1(t)$ exists for $R(t) > b - 1$ where b is such that $f_1(x)\, x^b$ is bounded as $x \to 0$.

The inversion formula gives

$$f_1(e^{-u})\,e^{-u} = \frac{1}{2\pi i}\int_{b'+1-i\infty}^{b'+1+i\infty} e^{ut}\,\Psi_1(t)\,dt \qquad b' > b$$

so that

$$f_1(x) = \frac{1}{2\pi i}\int_{b'-i\infty}^{b'+i\infty} x^{-t}\,\Psi_1(t-1)\,dt$$

Similarly we find that

$$f_2(x) = \frac{1}{2\pi i}\int_{b'-i\infty}^{b'+i\infty} x^{-t}\,\Psi_2(t-1)\,dt$$

where $b' < C$ and C is such that $f_2(x)\,x^C$ is bounded as $x \to \infty$.

Thus if $b < b' < C$ we may add the above results obtaining

$$f(x) = \frac{1}{2\pi i}\int_{b'-i\infty}^{b'+i\infty} x^{-t}\,\Psi(t-1)\,dt.$$

We thus see that the practical problem is the evaluation of integrals of the following form:

$$f(m) = 1/2\pi i\int_{b-i\infty}^{b+i\infty} e^{mz}\,\varphi(z)\,d(z) \tag{1}$$

Assuming the reality of $f(m)$ we find that φ is such that

$$\varphi(\bar{z}) = \overline{\varphi(z)}.$$

Also we may assume that $m > 0$. If $m < 0$ we have

$$f(m) = \frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} e^{-mz}\,\varphi(-z)\,dz$$

CHOOSING A DIFFERENT CONTOUR

In many cases φ exists not only to the right of the line $R(z) = b$ but at all points in the complex plane except at a number of poles. When the poles are simple or of low order, the evaluation by the summation of residues is straight forward.

However, many cases arise where the function $\varphi(z)$ exists almost everywhere but where the singularities are of a complicated type such as essential singularities. In particular such cases arise where we also know that all poles are on the real axis. The method to be described is applicable to such cases and to others where the poles are not on the axis but are sufficiently close to it.

An integral of the form (1) is often slowly convergent, the exponential factor merely introducing an oscillation to the intergrand.

If we choose a contour which bends towards the negative half-plane we introduce an exponentially decreasing factor which may speed up the overall convergence of the integral.

For convenience we try a parabola of the form:

$$y^2 = 4a(b - x), \qquad \text{where } x + iy = z.$$

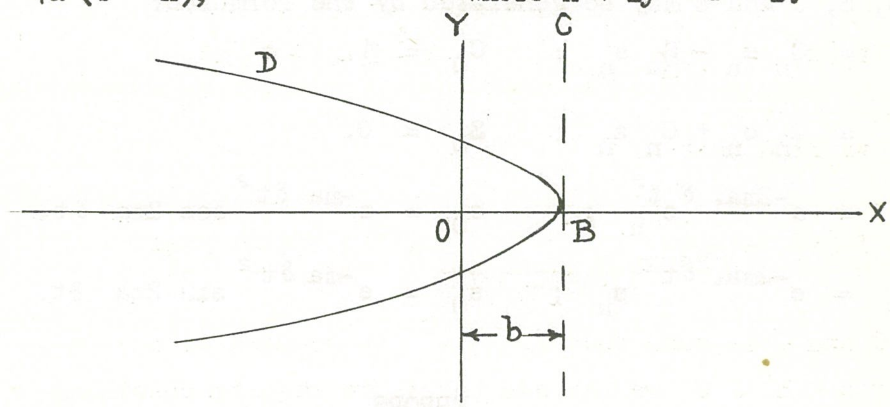

Figure 1.

If we have that in the region BCD of Figure 1, $\varphi(z)$ is analytic and that $|\varphi(z) z^k|$ is bounded for some positive k for z in this region we can easily show (see appendix) that the integral around the parabolic contour equals the integral along the line.

Write the equation of the parabola in parametric co-ordinates

$$y = 2at, \; x = b - at^2$$

and write

$$\varphi(z) = X(z) + i\,Y(z).$$

Then we find

$$f(m) = 2ae^{mb}/\pi \int_0^\infty e^{-mat^2} \begin{array}{l}(\cos 2mat - t \sin 2mat)X(z) \\ -(\sin 2mat + t \cos 2mat)Y(z)\end{array} \quad dz \qquad (2)$$

SPEEDING UP THE CALCULATION

The exponential and circular functions required for the numerical integration of (2) may be generated by a recursive method without great loss of accuracy and with much saving in time.

Write

$$C_n = e^{-mat^2} \cos 2mat$$

$$S_n = e^{-mat^2} \sin 2mat$$

where $t = n.\,\delta t$ and δt is the interval between points in the numerical integration.

Also write

$$c_n = e^{-2mat.\ \delta t\ -\ ma.\ \delta t^2} \cos 2ma\ \ \delta t$$

$$s_n = e^{-2mat.\ \delta t\ -\ ma\ \ \delta t^2} \sin 2ma\ \ \delta t.$$

Then C, S, c and s may be generated by the formulae:

$$C_{n+1} = C_n c_n - S_n s_n \ ; \qquad C_0 = 1.$$

$$S_{n+1} = S_n c_n + C_n s_n \ ; \qquad S_0 = 0.$$

$$c_{n+1} = e^{-2ma.\ \delta t^2} c_n \ ; \qquad c_0 = e^{-ma\ \delta t^2} \cos 2ma\ \ \delta t.$$

$$s_{n+1} = e^{-2ma.\ \delta t^2} s_n \ ; \qquad s_0 = e^{-ma\ \delta t^2} \sin 2ma\ \ \delta t.$$

ERRORS

The error in the calculation of C_n and S_n by using the recursive formulae will be partly due to the values of the constants c_0, s_0 and $e^{-2ma.\ \delta t^2}$ being inaccurate and partly due to the rounding off errors in each multiplication.

We consider the first source of error first. We find that

$$C_n + iS_n = (c_0 + is_0)^n \left(e^{-2ma\ \ \delta t^2}\right)^{\frac{n(n-1)}{2}}$$

therefore

$$\frac{\Delta(C_n + iS_n)}{C_n + iS_n} = \frac{n\Delta(c_0 + is_0)}{c_0 + is_0} + \frac{n(n-1)}{2}\ \frac{\Delta(e^{-2ma\ \ \delta t^2})}{e^{-2ma\ \ \delta t^2}}$$

where ΔC_n is the error in C_n etc.,

therefore

$$\left|\Delta C_n\right| < e^{-2ma.\ \delta t^2} \frac{n}{2}^2 \quad \frac{n}{2}^2 + (\sqrt{2} - 1/2)n \quad \epsilon_1$$

with a similar inequality for $\left|\Delta S_n\right|$,

where ϵ_1 is the maximum error in c_0, s_0 and $e^{-2ma.\ \delta t^2}$.

Let ϵ_2 be the maximum error in rounding off multiplications. We find that the maximum error due to this in C_n and S_n is less than

$$2\epsilon_2 + e^{-2ma\ \ \delta t^2} \left(\frac{n-1}{2}\right)^2 (n-1)\ \epsilon_2 .$$

So that the greatest error in C_n and S_n due to all sources is less than

$$e^{-2ma.\ \delta t^2}\ \frac{n^2}{2}\left\{\frac{n^2}{2}+(\sqrt{2}-1/2)n\right\}\epsilon_1$$

$$+\left\{2+(n-1)e^{-2ma\ \ \delta t^2}\ \frac{(n-1)^2}{2}\right\}\epsilon_2.$$

It may happen that the second error may be neglected compared with the first. The first error term reaches a maximum approximately where

$$n = 1/\sqrt{ma\ \delta t^2} = n_1 \quad \text{say.}$$

Evaluating at this point, we find the error is less than

$$\frac{\epsilon_1}{e}\left\{\frac{n_1^2}{2}+(\sqrt{2}-\frac{1}{2})\,n_1\right\}$$

For the values $m = a = 1$, $\delta t = 1/4$ we find $n_1 = 4$ and the maximum error equals $4\epsilon_1$ approximately. This was checked on a desk calculator working always to an accuracy of 6 decimal places with values of $e^{-2ma.\ \delta t^2}$, c_0 and s_0 correct to this accuracy. The error at the maximum should be less than 2×10^{-6}.

It was found that the largest error in C_n was at $n = 6$, with an error of $.6 \times 10^{-6}$ and the largest in S_n at $n = 3$ with an error of 1.5×10^{-6}.

As the error due to the integration formula used depends on the nature of the function $\varphi(z)$, we do not discuss errors from this source.

CHOICE OF PARAMETERS

The constants a, b and t are free to be chosen in the most suitable manner for the particular problem. As the sum which represents the integral in equation (2) has to be multiplied by e^{mb} we see that e^{mb} must be as low as other considerations will allow or otherwise the sum will have a low value, and hence the product which represents $f(m)$ will be inaccurate. The exact choice of b will be governed by this principle and by questions of scaling.

For the best choice of a we must again keep in mind what values are suitable to enable X and Y to be suitably scaled. Apart from this, a and δt should be chosen so that a compromise is reached between having too large an error due to the recursion formula ($ma\ \delta t^2$ too small) and having large errors due to the integration formula used. The exact choice depends on the value of m and on the nature of the function φ. It should be noted that the number of points that must be taken before the integral dies away to e^{-k} is $n_1\sqrt{k}$, as this will affect the time taken to perform the integration.

SILLIAC PROGRAM

A program has been prepared for SILLIAC, using this method. The user must write a subroutine which will compute the real and imaginary parts of the function $\varphi(z)$ scaled in any manner, for given values of the real and imaginary parts of the argument z. The trapezoidal rule is used for the interpolation.

For the function $\varphi = 1/z$, the correct result is 1. This was tried with the SILLIAC program with the values $a = b = m = 1$ and $\delta t = 1/4$. The number of points that had to be taken before the integrand died away was 20 and the result was correct to within $.5 \times 10^{-9}$.

APPENDIX

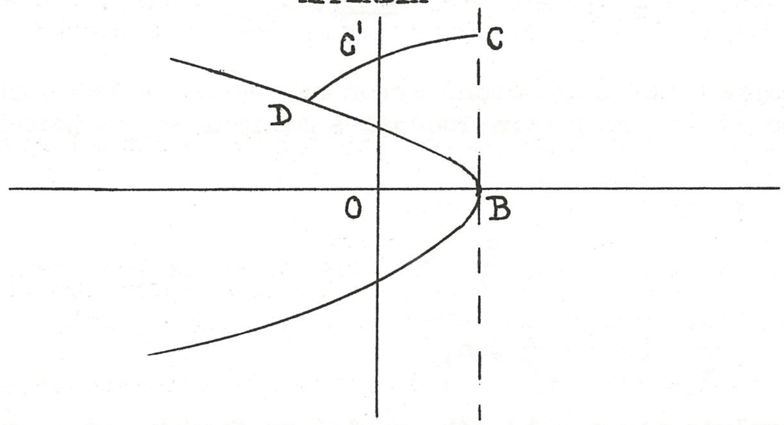

Figure 2.

Let CC'D (Figure 2) be part of a circle with centre 0 and radius R. We need only show that the integral around CD tends to zero as $R \to \infty$

$$\int_{CD} e^{mz}\,\varphi(z)\,dz = \int_{CC'} e^{mz}\,\varphi(z)\,dz + \int_{C'D} e^{mz}\,\varphi(z)\,dz$$

$$= I_1 + I_2 \quad \text{say.}$$

The length of arc CC' is no more than $\pi/2\,b$. Also $\varphi(z) < AR^{-k}$, for some A, therefore,

$$|I_1| \leqslant e^{mb}.AR^{-k}\,\pi/2\,b. \to 0 \text{ as } R \to \infty$$

Writing

$z = R\,\mathrm{cis}\,(\theta + \pi/2)$, we have:

$$|I_2| \leqslant \int_0^{\pi/2} e^{mR\,\mathrm{cis}\,(\theta + \pi/2)}\,AR^{-k}\,R\,d\theta$$

$$= AR^{-k+1}\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta$$

$$\leqslant AR^{-k+1}\int_0^{\pi/2} e^{-mR\;2\theta/\pi}\,d\theta$$

$$= \frac{\pi A}{2m}\,R^{-k}\,(1 - e^{-mR}) \to 0.$$

REFERENCE: 1. "Operational Methods in Applied Mathematics", 2nd Ed. p.72. By H.S. Carslow and J.C. Jaegar.

DISCUSSION

Dr. M.V. Wilkes, University of Cambridge.

Do I understand you correctly? When you work out the integral you have real and imaginary parts. As a result you should get real and imaginary parts in the result. Presumably you only require the real part of the result so you only work out this part, but would not computation of the imaginary part come out as a check?

Mr. J.C. Butcher (In Reply)

I have looked at this point, but for the minute I have forgotten what I decided.

Dr. M.V. Wilkes, University of Cambridge.

There is a point. Perhaps the imaginary part vanishes by symmetry. If that is the case it wouldn't be a very good check.

Mr. J.P.O. Silberstein, Aeronautical Research Laboratories, Melbourne.

I think what happens is that the part of the integral below the real axis is the conjugate of the integral above.

Dr. M.V. Wilkes, University of Cambridge.

Suppose you used a curve that is not symmetric. Then the imaginary part would not vanish by just being the conjugate of the real part. You may then succeed to get a check this way.

Dr. T. Pearcy, C.S.I.R.O., Sydney.

I presume you don't approach through singular points.

Mr. J.C. Butcher (In Reply)

One of the conditions under which one can use parabolas is that all poles fall outside of it. In the case I mentioned all poles fall on the real axis. I think a lot of cases arise where this does happen or at least where you can ensure that they are on the real axis.

Dr. T. Pearcy, C.S.I.R.O., Sydney.

In that case would it be possible to optimise your choice of parabola.

Mr. J.C. Butcher (In Reply)

It probably is. It has been suggested that I look at functions other than a parabola.

Dr. J.M. Bennett, University of Sydney.

Wouldn't you choose a contour for integration such that the expression you are integrating diminished as quickly as possible?

Mr. J.C. Butcher (In Reply)

If the integrand dimishes too quickly the values you are evaluating are too far apart and you get a rapidly varying function.

Mr. J.P.O. Siberstein, Aeronautical Research Laboratories, Melbourne.

How much space is there left to evaluate the function in the SILLIAC.

Mr. J.C. Butcher (In Reply)

The programme is 50 words which leaves about 900 to evaluate the function.

Dr. M.V. Wilkes, University of Cambridge.

Have you checked against integrals that are well known?

Mr. J.C. Butcher (In Reply)

Yes, I think 1/Z. The results were so good I suspected that an identity was involved. There were 53 points of integration involved and a trapezoidal rule was used since I could not see any advantage in using a Simpson. Some people at Sydney have tried to see that the journal would be out in this case.

Professor T.M. Cherry, University of Melbourne.

Has anything been done on a line suggested by this work? For this particular case you would choose the starting point value of b where the integrand is a minimum on the real axis. There would be a pole somewhere on the left and you would come down from that pole and, taking the whole integrand, you would go up to infinity with the exponential factor. Now, starting from the minimum you proceed at right angles and you might aim at using the method of steepest descents running almost down the steepest line on the contours of modulus of the integrand. I was going to ask whether you had actually done this on a computer, i.e. at each step finding what is the most favourable direction to take so that somewhere near optimum increments to the integral can be found.

Dr. T. Pearcy, C.S.I.R.O., Sydney.

I think this was tried on C.S.I.R.A.C. about three years ago but I have forgotten the details.